ON A METHOD OF SOLUTION OF THE NONLINEAR PROBLEMS OF STABILITY OF DEFORMABLE SYSTEMS

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Variational methods play an important part in the solution of nonlinear problems of stability.

Numerous approximate solutions have been obtained for nonlinear stability. In the qualitative aspects they do not, as a rule, raise any doubts. Using variational methods, it is usually possible to establish the existence of critical states and the non-uniqueness of the forms of equilibrium. In the quantitative estimates, however, the results obtained for nonlinear regions differ considerably, depending on the method of analysis and the procedure for approximation. This can be explained by the fact that, at large displacements, the shape of the deformed surface of a shell changes so much that it cannot be described by functions containing only one, or even two, variable parameters.

Introduction of a larger number of parameters causes serious computational difficulties. The problem leads, as a rule, to the solution of a system of cubic equations, whose number is equal to the number of variable parameters. The analysis of possible forms of equilibrium requires the determination of the real roots of these equations depending on external forces. In practice, this problem proves to be so cumbersome that it is necessary to assume only one or, at most, two variable parameters. Sometives it is possible to introduce a larger number of parameters. This can be achieved usually in the cases where some of the equations prove to be linear or are artificially linearized. Especially difficult are investigations of systems subjected to several independently varying forces.

The technical aspect of the problem is to some extent simplified if high-speed electronic digital computers are used. Nevertheless, a significant step toward the increase of the number of variable parameters has yet to be made.

In the following, a possibility is considered of increasing the number of variable parameters and developing most efficient methods of solution, not in the sense of formal convergence, but of practical usefulness.

The effectiveness of an approximate method may be evaluated by comparing with the exact solution. Since, at the present time, exact solutions of nonlinear problems of the theory of shells are not available, the first part of this paper deals with the problem of axially-symmetric bending of a shallow spherical dome. It is solved by a numerical method and the obtained solution is assumed as the standard.

It will be assumed that the use of electronic digital computers is an organic part of the work. The algorithm of computation is given.

1. The equations for a shallow spherical shell [1], subjected to a uniform pressure p acting on the convex side, have the form

$$\rho\psi'' + \psi' - \frac{\psi}{\rho} = \vartheta\left(\theta_k\rho + \frac{\vartheta}{2}\right), \qquad \rho\vartheta'' + \vartheta' - \frac{\vartheta}{\rho} = -\lambda\psi\left(\theta_k\rho + \vartheta\right) + \nu\rho^2 \tag{1.1}$$

$$\left(\rho = \frac{r}{a},\ \psi = -\frac{T_1\rho}{Eh},\ \lambda = 12(1-\mu^2)\frac{a^2}{h^2},\ \nu = \lambda\frac{pa}{2Eh}\right)$$

Here, ρ is the dimensionless radius, which will be assumed as the independent variable; r is the running radius; a is the radius of the outer contour (Fig. 1); T_1 is the radial tensile force; h is the thickness of the shell; ϑ is the angle of rotation of the normal; θ_k is the slope of the undeformed shell at $r = a$.

Introducing the rise of the dome H as a parameter, we transform the equations to the following form

$$\rho\Psi'' + \Psi' - \frac{\Psi}{\rho} = \Theta\left(\frac{2H}{h}\rho + \frac{1}{2}\Theta\right) \tag{1.2}$$

$$\rho\Theta'' + \Theta' - \frac{\Theta}{\rho} = -12(1-\mu^2)\Psi\left(\frac{2H}{h}\rho + \Theta\right) + 6(1-\mu^2)p_0\rho^2 \tag{1.3}$$

where

$$\Theta = \frac{a}{h}\vartheta, \qquad \Psi = \frac{a^2}{h^2}\psi, \qquad p_0 = \frac{pa^4}{Eh^4}$$

The axial displacement w is

$$\frac{w}{h} = \int_{\rho}^{1} \Theta \, d\rho \tag{1.4}$$

We introduce the notation

$$\Theta' = u, \qquad \Psi' = v \tag{1.5}$$

and we rewrite equation (1.2) in finite differences

$$\Delta\Theta = u\Delta\rho, \qquad \Delta\Psi = v\Delta\rho \tag{1.6}$$

$$\Delta v = \left(-\frac{v}{\rho} + \frac{\Psi}{\rho^2} + \frac{2H}{h}\Theta + \frac{1}{2}\frac{\Theta^2}{\rho}\right)\Delta\rho \tag{1.7}$$

$$\Delta u = \left(-\frac{u}{\rho} + \frac{\Theta}{\rho^2} - 10.92\frac{2H}{h}\Psi - 10.92\frac{\Psi\Theta}{\rho} + 5.46p_0\rho\right)\Delta\rho$$

Here and in the following we assume $\mu = 0.3$.

For $\rho = 0$, the functions Θ and Ψ become equal to zero.

Having given the values $u = u_0$ and $v = v_0$, we find $\Delta\Theta$ and $\Delta\Psi$ from equations (1.6), and then we determine Δu and Δv from equations (1.7). Adding the increments of the functions to the preceding values, we continue the integration procedure up to the value $\rho = 1$. On the contour, the conditions of fully fixed edges are to be satisfied, i.e. $\Theta_{\rho=1} = 0$ and

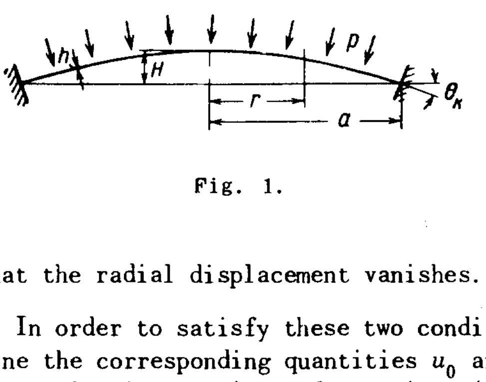

Fig. 1.

$$Z_k = Z_{\rho=1} = \left| \Psi' - \mu\frac{\Psi}{\rho} \right|_{\rho=1} = 0$$

The second condition means that the radial displacement vanishes.

In order to satisfy these two conditions it is necessary to determine the corresponding quantities u_0 and v_0. This is accomplished by successive integrations of equations (1.7) and analysis of the resulting quantities $\Theta_{\rho=1} = \Theta_k$ and $Z_{\rho=1} = Z_k$.

At first, to obtain a rough estimate, for the given values of the parameters $2H/h$ and p_0 we perform many integrations for various values of u_0 and v_0. In this way we obtain two curves in the u_0v_0-plane (Fig. 2). One of these curves corresponds to the values of u_0 and v_0 for which the first boundary condition ($\Theta_k = 0$) is satisfied; the second curve corresponds to the condition $Z_k = 0$. The points of intersection of the two curves give the sought values of u_0 and v_0. The number of the points of intersection is equal to the number of the forms of equilibrium for a given pressure p_0. In Fig. 2, these points are denoted by *1*, *2* and *3*, in the planes of the variables u_0, v_0 and p_0, w_0/h.

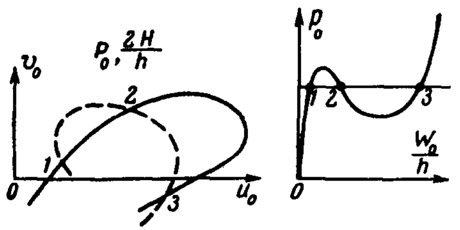

Fig. 2.

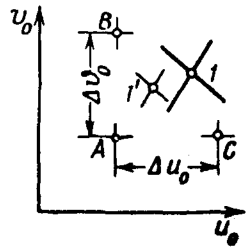

Fig. 3.

The refined values of u_0 and v_0 for each form of equilibrium may be found by successive linear interpolations.

Let us select, for example, in the vicinity of the point *1* three arbitrary points A, B and C (Fig. 3). Integrating equations (1.7) we find Θ_k and Z_k for each of these points. Having three values Θ_k and three values Z_k, we can construct two planes, $\Theta_k = \Theta_k(u_0, v_0)$ and $Z_k = Z_k(u_0, v_0)$. The point of intersection of the planes Θ_k and Z_k with the coordinate plane is the first approximation *1'* to the sought point *1*. Next, we move the point A to the point *1'* and reduce Δu_0 and Δv_0 several times, and we continue the process of iteration until the differences between the coordinates of two successive approximations becomes smaller than a given value.

In the performed calculations this difference was equal to 0.0001, which represents about 0.01% of the corresponding values of u_0 and v_0. The intervals of integration were equal to 0.01 for the preliminary calculations, and equal to 0.001 for the refined calculations. At the boundary of the shell, where the curvature changes rapidly, the intervals were reduced to 0.0001. Further reduction of the intervals is of no practical effect.

In order to avoid a complicated investigation of the u_0v_0-plane, the point *1* was determined for the increasing values of p_0, starting with $p_0 = 0$.

As p_0 changes, the point *1* moves. If the parameter p_0 is increased by a small value, the point *1* does not move considerably and can be followed up by the above method of linear interpolation. In this way, the locations of the points *1*, *2* and *3* in the plane u_0v_0 are determined as functions of p_0.

Near the critical states, the points *1* and *2* (or *2* and *3*) approach each other and, finally, coincide. The linear interpolation becomes then inefficient and we resort to the quadratic interpolation. In the region where the points *1* and *2* (or *2* and *3*) are close, a "trap" is constructed using nine points (Fig. 4). The curves $\Theta_k = 0$ and $Z_k = 0$ become second-order parabolas. Varying the quantities u_0 and v_0 and the values of the parameter p_0, and using the criterion of multiple roots, it is possible to determine the upper and lower critical pressures.

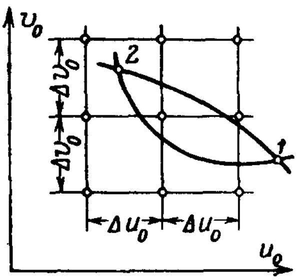

Fig. 4.

The loci of the points in the u_0v_0-plane are shown in Fig. 5, and the relation between the pressure p_0 and the deflection w_0/h is shown in Fig. 6.

Varying the parameter $2H/h$ and using a nine point "trap" in the plane u_0v_0, we can determine the relation between the critical pressure and the rise of the dome (Fig. 7).

During final stages of the preparation of this paper, [2] was published with the solution of the same problem by a different algorithm. Both results coincide.

2. We note the following essential property characterizing the majority of the problems of a similar type. It is evident that, during large deflections, the form of the elastic surface of the shell is subjected to considerable changes which cannot be taken into account by one or two parameters used in the approximating functions.

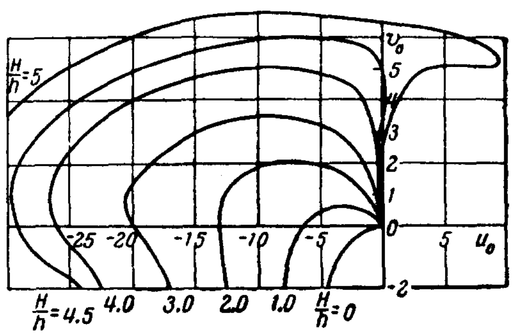

Fig. 5.

In the case of a shallow spherical shell, this point may be demonstrated by assuming, as the first approximation, the deflections in the same form as for a circular plate

$$\frac{w}{h} = C(1-\rho^2)^2 \qquad (2.1)$$

where C is the only variable parameter.

Using the energy method or Galerkin's method, the critical relation between pressure and deflection may be obtained. Comparison of the diagrams (Fig. 8) obtained earlier in [3] and the results of the numerical analysis (continuous lines in Fig. 8) shows that, in this case, the variation of solely one parameter gives a sufficient accuracy only for the domes with relatively small rise H. For the rise $H = 4h$, the discrepancy in the magnitudes of critical pressures proves to be so large that the approximate solution loses any significance.

This discrepancy can be explained by the fact that the function (2.1) does not reflect the actual form of the elastic surface of a shell. The curves shown in Fig. 5 indicate that for the rise $H > 4.5h$ the quantity u_0 assumes positive values. This means that, in the first stage of leading, the curvature of the dome increases in the center, and the elastic line of the meridional arc (Fig. 9) has the shape of the curve a, which is considerably different from the curve b given by the expression (2.1). The existence of the central convexity is reflected in the configuration of the curves in Fig. 10, which give the relation between p_0 and w_0/h for relatively large

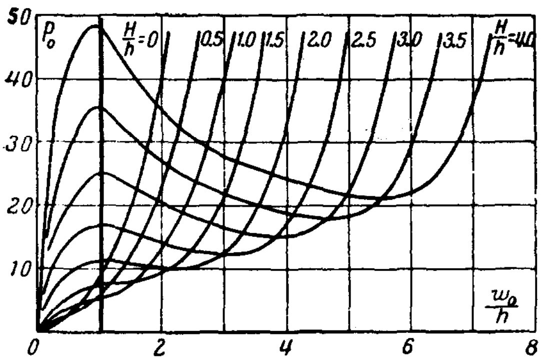

Fig. 6.

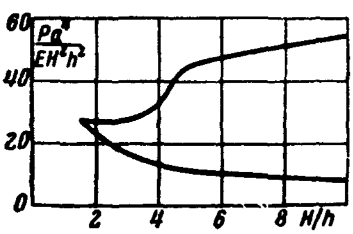

Fig. 7.

values H/h. The center of the shell moves initially in the direction of loading, and then a small reverse displacement appears. In certain intervals of p_0 there are not three but five forms of equilibrium (the extreme right-hand branch is not shown in Fig. 10). For still larger values, the development of new forms of equilibrium is possible. It is obvious that these phenomena cannot be included within the scope of the first approximation with one parameter. The use of a larger number of parameters meets unmanageable computational difficulties.

In the example considered, there is no need to resort to higher approximations, and to approximate solutions in general, because we have the exact solution. However, the numerical solutions of the problems of similar type can be expected only in the cases of ordinary differential equations, while the majority of more important and interesting problems of stability of shells reduces to the solution of partial differential equations, where the use of digital computers proves to be ineffective.

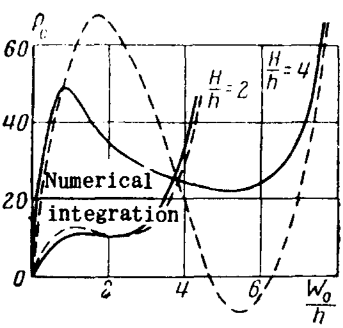

Fig. 8.

The question arises, what should be done in more complicated problems In particular, how to approach the

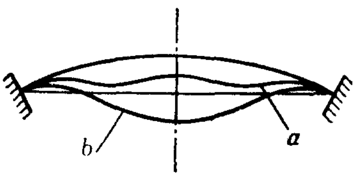

Fig. 9.

same problem of a spherical dome, but for the conditions of unsymmetrical forms of instability.

Up to date, in many papers the attempt has been made to use a digital computer for the solution of problems by the variational method and, thus, to increase the number of variable parameters. In this, the usual approach is maintained, but the computational possibilities are enlarged.

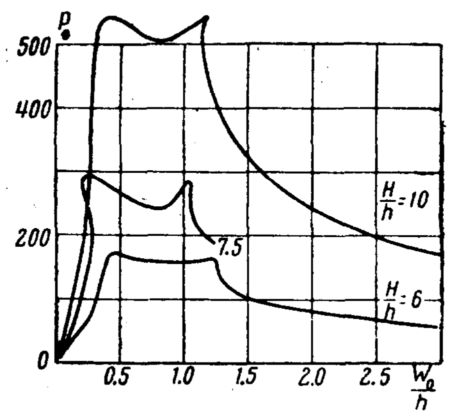

Fig. 10.

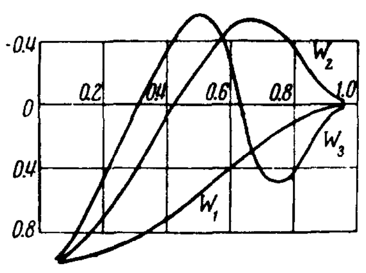

Fig. 11.

Such a method, however, does not give significant results. The solution reduces again to a system of complicated, nonlinear algebraic equations. The construction of an algorithm for their solution (with the non-uniqueness taken into account) represents often an insoluble problem.

The analysis of the process in time proves to be a more efficient approach to the solution of similar problems. The introduction of one more independent variable (time) for the investigation of statically loaded systems leads, paradoxically, not to a complication, but to a simplification of the problem, provided the computational techniques are properly utilized.

We shall consider again a shallow spherical shell, and with this example we shall explain the above statement.

3. We shall limit ourselves (for the sake of simplicity) to the symmetrical forms of deformation of the shell. We rewrite equations (1.2) introducing the inertial forces and the terms of linear damping

$$\rho\Psi'' + \Psi' - \frac{\Psi}{\rho} = \Theta\left(\frac{2H}{h}\rho + \frac{1}{2}\Theta\right) \tag{3.1}$$

$$\rho\Theta'' + \Theta' - \frac{\Theta}{\rho} = -12(1-\mu^2)\Psi\left(\frac{2H}{h}\rho + \Theta\right) + 6(1-\mu^2)p_0\rho^2 + \\ + \varkappa\int_0^\rho \dot{W}\rho d\rho + 12(1-\mu^2)\int_0^\rho \ddot{W}\rho d\rho \tag{3.2}$$

$$\tau = t\sqrt{\frac{gE}{\gamma a^2}} \tag{3.3}$$

Here, $W = w/\rho$ represents the dimensionless axial displacement, κ is a certain, as yet undetermined coefficient of damping, γ is the specific weight of the material of the shell. The dots denote differentiation with respect to the dimensionless time τ. We assume that

$$W = A_1W_1 + A_2W_2 + A_3W_3 \tag{3.4}$$

where A_1, A_2, A_3 are certain parameters depending on time

$$W_1 = (1-\rho^2)^2, \qquad W_2 = (1-\rho^2)^2(1-6\rho^2) \\ W_3 = (1-\rho^2)^2(1-14\rho^2+28\rho^4) \tag{3.5}$$

The curves corresponding to these functions are shown in Fig. 11. The function Θ is defined as the derivative of W, i.e.

$$\Theta = A_1\Theta_1 + A_2\Theta_2 + A_3\Theta_3 \\ \Theta_1 = 4(-\rho+\rho^3), \qquad \Theta_2 = 4(-4\rho+13\rho^3-9\rho^5) \\ \Theta_3 = 4(-8\rho+57\rho^3-105\rho^5+56\rho^7) \tag{3.6}$$

From equation (3.1) we find

$$\Psi = \frac{2H}{h}(A_1\Psi_1 + A_2\Psi_2 + A_3\Psi_3) + A_1^2\Psi_{11} + A_2^2\Psi_{22} + \\ + A_3^2\Psi_{33} + A_1A_2\Psi_{12} + A_1A_3\Psi_{13} + A_2A_3\Psi_{23} \tag{3.7}$$

$$\Psi_1 = 4\left(a_1\rho - \tfrac{1}{8}\rho^3 + \tfrac{1}{24}\rho^5\right)$$

$$\Psi_2 = 4\left(a_2\rho - \tfrac{1}{2}\rho^3 + \tfrac{13}{24}\rho^5 - \tfrac{3}{16}\rho^7\right)$$
$$\Psi_3 = 4\left(a_3\rho - \rho^3 + \tfrac{19}{8}\rho^5 - \tfrac{35}{16}\rho^7 + \tfrac{7}{10}\rho^9\right)$$
$$\Psi_{11} = 8\left(a_{11}\rho + \tfrac{1}{8}\rho^3 - \tfrac{1}{12}\rho^5 + \tfrac{1}{48}\rho^7\right)$$
$$\Psi_{22} = 8\left(a_{22}\rho + 2\rho^3 - \tfrac{13}{3}\rho^5 + \tfrac{241}{48}\rho^7 - \tfrac{117}{40}\rho^9 + \tfrac{27}{40}\rho^{11}\right)$$
$$\Psi_{33} = 8\left(a_{33}\rho + 8\rho^3 - 38\rho^5 + \tfrac{1643}{16}\rho^7 - \tfrac{6433}{40}\rho^9 + \tfrac{5803}{40}\rho^{11} - 70\rho^{13} + 14\rho^{15}\right)$$
$$\Psi_{12} = 16\left(a_{12}\rho + \tfrac{1}{2}\rho^3 - \tfrac{17}{24}\rho^5 + \tfrac{11}{24}\rho^7 - \tfrac{9}{80}\rho^9\right)$$
$$\Psi_{13} = 16\left(a_{13}\rho + \rho^3 - \tfrac{65}{24}\rho^5 + \tfrac{27}{8}\rho^7 - \tfrac{161}{80}\rho^9 + \tfrac{7}{15}\rho^{11}\right)$$
$$\Psi_{23} = 16\left(a_{23}\rho + 4\rho^3 - \tfrac{83}{6}\rho^5 + \tfrac{411}{16}\rho^7 - \tfrac{1051}{40}\rho^9 + \tfrac{1673}{120}\rho^{11} - 3\rho^{13}\right)$$

The constants a_1, a_2, a_3, a_{11}, ..., are determined by the condition

$$\left|\Psi' - \mu\frac{\Psi}{\rho}\right|_{\rho=1} = 0$$

Substituting the expressions (3.6) and (3.7) for Θ and Ψ into equation (3.2), multiplying all the terms by Θ_1, Θ_2 and Θ_3, and integrating with respect to ρ from 0 to 1, we obtain the following three equations

$$\begin{aligned} \ddot{A}_1 + \varkappa\dot{A}_1 + L_1(A_1, A_2, A_3) &= 0 \\ \ddot{A}_2 + \varkappa\dot{A}_2 + L_2(A_1, A_2, A_3) &= 0 \\ \ddot{A}_3 + \varkappa\dot{A}_3 + L_3(A_1, A_2, A_3) &= 0 \end{aligned} \tag{3.8}$$

where L_1, L_2 and L_3 are certain polynomials of the third order of A_1, A_2 and A_3.

For example

$$L_1(A_1, A_2, A_3) = -1.6667p_0 + 9.76801A_1 -$$
$$-4.88400A_2 + 2.9304A_3 + \left(\frac{2H}{h}\right)^2(2.03175A_1 - 0.51587A_2 + 0.30952A_3) +$$
$$+\frac{2H}{h}(-6.4286A_1^2 - 8.71428A_2^2 - 14.9619A_3^2 - 4.4286A_1A_2 + 2.3651A_1A_3 -$$
$$-4.88254A_2A_3) + 4.60317A_1^3 + 13.54603A_2^3 - 4.12394A_3^3 + 9.38095A_1^2A_2 - 4.07937A_1^2A_3 +$$
$$+24.1524A_2^2A_1 + 13.5535A_2^2A_3 + 35.5832A_3^2A_1 + 31.6707A_3^2A_2 + 10.3365A_1A_2A_3 = 0$$

The expressions for L_2 and L_3 have exactly the same forms with different values of the numerical coefficients.

The functions W_1, W_2 and W_3, given by (3.5), are such that

$$\int_0^1 \left[\int_0^\rho W_i \rho d\rho\right] \theta_j \, d\rho = 0 \qquad \text{for } i \neq j \tag{3.9}$$

Therefore, the derivatives $\ddot{A}_1$ and $\dot{A}_1$ appear only in the first, the derivatives $\ddot{A}_2$ and $\dot{A}_2$ only in the second, and the derivatives $\ddot{A}_3$ and $\dot{A}_3$ only in the third of equations (3.8).

Should there be a necessity of the fourth approximation, the function W_4 would have to be of the form

$$W_4 = (1 - \rho^2)^2 (1 + a\rho^2 + b\rho^4 + c\rho^6)$$

with the coefficients a, b and c determined from the orthogonality condition (3.9).

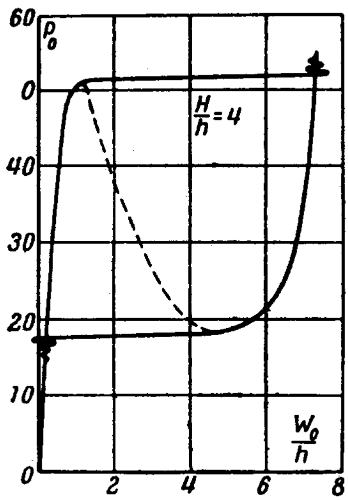

Fig. 12.

The most tedious operation is the calculation of the coefficients of the polynomials L_1, L_2 and L_3. This part of the work, however, can be easily performed by computers. Therefore, in this problem the number of parameters A_i could have been increased.

We now integrate equations (3.8). Denoting

$$\dot{A}_1 = u_1, \qquad \dot{A}_2 = u_2, \qquad \dot{A}_3 = u_3$$

we have

$$\begin{gathered} \Delta A_1 = u_1 \Delta\tau, \qquad \Delta A_2 = u_2 \Delta\tau, \\ \Delta A_3 = u_3 \Delta\tau, \ \Delta u_1 = (-\varkappa u_1 - L_1)\,\Delta\tau \\ \Delta u_2 = (-\varkappa u_2 - L_2)\,\Delta\tau \\ \Delta u_3 = (-\varkappa u_3 - L_3)\,\Delta\tau \end{gathered} \tag{3.10}$$

We assume that p_0 is a given function of time. Let $p_0 = K\tau$ where K is a constant quantity which should be assumed sufficiently small, for a given problem, in order to consider the loading as static.

The selection of K and $\Delta\tau$ can be facilitated through the estimate of the order of magnitude of the natural frequencies of the shell. This estimate can easily be obtained if the nonlinear components in the expressions L_1, L_2 and L_3 are neglected.

In these calculations, $K = 1$ and $\Delta\tau = 0.001$ have been assumed.

At the initial instant of time (for $\tau = 0$)

$$u_1 = u_2 = u_3 = 0, \qquad A_1 = A_2 = A_3 = 0$$

Next, from equations (3.10) we calculate successively

$$\Delta A_1, \ \Delta A_2, \ \Delta A_3, \ A_1, \ A_2, \ A_3, \ \Delta u_1, \ \Delta u_2, \ \Delta u_3 \quad \text{etc.}$$

and we construct the relation between the pressure and the deflection. This relation is shown in Figs. 12 and 13, for $H/h = 4$ and $H/h = 8$.

At the first critical pressure, the deflection increases rapidly (snap-through), accompanied by oscillations around a new position of equilibrium. For a fuller reproduction of the physical aspect of the phenomenon the linear damping has been introduced into the equations. The coefficient of damping κ has been selected from the point of instructiveness; not too large, in order to retain the oscillatory character of the process, and not too small, in order to damp out the oscillations sufficiently fast. In these calculations $\kappa = 3$ has been assumed.

During unloading, the reverse snapping appears, accompanied also by oscillations.

The equations (3.8) have been integrated for the values $H/h = 4$, 6 and 8. The critical pressures obtained are marked by points in Fig. 14. The same figure contains sections of the curves from the diagram in Fig. 7. It is evident that the introduction of three parameters results in a sufficient accuracy of the determination of the critical pressures.

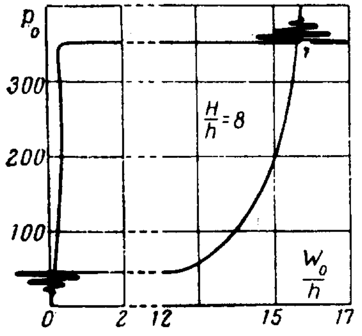

Fig. 13.

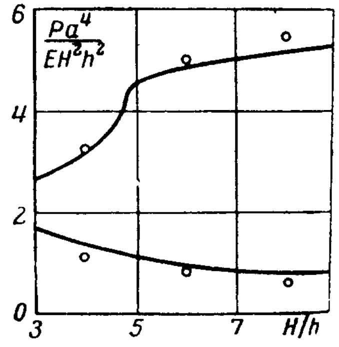

Fig. 14.

4. We shall consider now the question of the general evaluation of the proposed method.

Introducing the time factor, we obtain the single valuedness, determined by the history of the problem, and the nonlinearity of the problem practically does not cause any difficulties. This is evident, in particular, from the example of the numerical solution of equations (3.8).

Without the introduction of the time factor and applying the usual methods to the determination of the possible forms of equilibrium, the same system of equations (3.8) must be considered in a different manner. Assuming

$$\ddot{A}_1 = \ddot{A}_2 = \ddot{A}_3 = 0$$

we obtain the system of cubic equations

$$L_1(A_1, A_2, A_3) = 0, \qquad L_2(A_1, A_2, A_3) = 0, \qquad L_3(A_1, A_2, A_3) = 0$$

which have to be solved with respect to A_1, A_2 and A_3 for various values of p_0. With three parameters, this task becomes already more than complicated. With the number of parameters increasing, the difficulties grow progressively.

The proposed method is free from this disadvantage. The introduction of additional terms of the expansion increases only the preparatory work, but does not influence the numerical solution of the equations of motion, because a computer easily performs the operation of integration for an arbitrary number of the variable parameters.

In the proposed method, the tasks performed by a computer do not require large amounts of machine time: the calculation of the coefficients in the preparation of the equations and the integration of the equations with initial conditions. The most difficult problem for a machine, the boundary value problem, is solved by the variational method.

The scope of applicability of the method is, evidently, sufficiently broad. It is possible to include the problems whose solutions are determined by the history of loading of the system; for example, the problems of stability of plastically deformable bodies. To this category belongs the analysis of structures under conditions of non-proportional loading and thermal effects varying in time. The method presented is applicable to dynamical problems, which can be solved with no more effort than the static problems. In this, the non-uniqueness of the solution is removed by the continuous history of the development of the process.

Finally, of great importance is the fact that in the solution of practical problems the investigator is not bound in advance by a definite criterion of stability. The behavior of a structure in time may be controlled and evaluated in many different ways. In the course of analysis, the limit state may be assumed as the state corresponding to the appearance of plastic deformations, or the state corresponding to fast increasing displacements, or the state corresponding to the ultimate loading capacity when the deformations increase at decreasing loading, etc.

The strong and the weak points of the method reveal themselves further in the course of solution of specific problems.

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